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FAST TRACK COMMUNICATION

Universality classes of polymer melts and conformal sigma models

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Abstract
In the usual statistical model of a dense polymer (a single space-filling loop on a lattice) in two dimensions the loop does not cross itself. We modify this by including intersections in which *three* lines can cross at the same point, with some statistical weight w per crossing. We show that our model describes a line of critical theories with continuously varying exponents depending on w, described by a conformally invariant nonlinear sigma model with varying coupling constant $g_\sigma^2 \geqslant 0$. For the boundary critical behavior, or the model defined in a strip, we propose an exact formula for the ℓ-leg exponents, $h_\ell = g_\sigma^2 \ell(\ell - 2)/8$, which is shown numerically to hold very well.

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(Some figures in this article are in colour only in the electronic version)

Loop models are ubiquitous in low-dimensional statistical mechanics, and have been studied for decades [1]. They have recently grown to play a major role in topological quantum computing [2].

Most loop models studied so far have to forbid intersections to be solvable. Their critical exponents can then be calculated using techniques of conformal field theory (CFT) [3], Coulomb gas or stochastic Loewner evolution (SLE) [4, 5].

The role of intersections in these models is not too well understood in general. It has been studied in detail in the special case of 'dense' polymers where a small number of loops on a lattice are forced to occupy a finite fraction of the sites, and thus resemble a real polymer melt. In this case, allowing generic intersections (where two lines cross) does take the model to a universality class [6, 7] which is very different from the usual dense polymers. The properties of the melt thus obtained are however close to those of ordinary Brownian motion, and not very interesting.

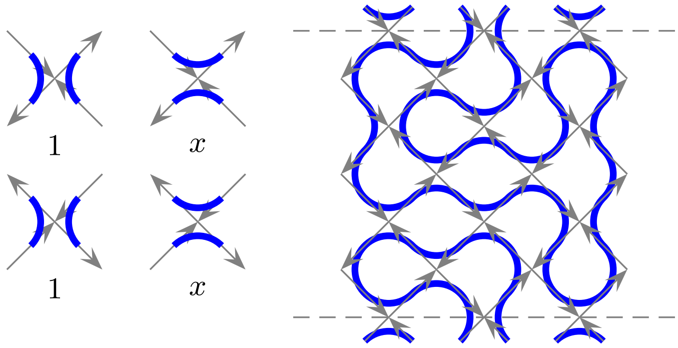

Figure 1. Vertices, weights and sample configuration for dense polymers on a square lattice of width $L = 3$. Boundary conditions are free in the horizontal (space) direction and periodic in the vertical (imaginary time) direction. The alternating $\square$, $\bar{\square}$ representations correspond to a lattice orientation, conserved along each loop.

A close look at critical exponents for dense polymers shows that, while the four-leg operator is relevant, the six-leg operator is marginal. The main result of this communication is that only allowing intersections where *three* lines cross simultaneously indeed produces a very different behavior: a line of critical points is obtained, with central charge $c = -2$, and continuously varying critical exponents. Moreover, this critical line is described by a nonlinear σ-model (see below)—a very unusual situation in statistical mechanics.

A convenient way to describe many loop models is to use a supersymmetric (SUSY) formulation, in which the degrees of freedom can take bosonic or fermionic values [8], and the action enjoys supergroup invariance. The last few years have witnessed an intense interest in such theories in the framework of the AdS/CFT conjecture [9, 10]. An archetypal example in this field is the principal chiral model (PCM) on PSL(2|2), which is expected to be conformal invariant for a large range of values of the coupling constant g_σ^2. This is very different from what happens in ordinary groups, such as $SU(2)$, where the PCM exhibits asymptotic freedom and spontaneous mass generation. Despite significant progress [11], σ-models on supergroups remain notoriously hard to solve.

We show in this communication that considering polymer melts with six-leg crossings leads to close cousins of the PSL$(n|n)$ models: σ-models on superprojective spaces (the super-analogs of ordinary projective spaces) $U(n|n)/U(n-1|n) \times U(1)$. This identification has crucial consequences. It bridges the study of loop models with the one of σ models; it gives direct access to the properties of the σ-models both numerically and, potentially, analytically using the techniques developed in [12, 13]. It also allows the determination of the critical exponents in the original geometrical problem.

Dense polymers and (SUSY) spin chains. The universality class of dense polymers is generically obtained when one forces a finite number of self-avoiding loops or walks to fill a fraction of space $\rho > 0$. The loop model in figure 1 is in the generic (i.e. ρ-independent) dense polymer universality class.

The CFT of dense polymers has $c = -2$; we review it below. It was discovered a few years back [6, 7] that if one allows four-leg crossings the model flows to a different universality class with $c = -1$, and trivial geometrical exponents. Such crossings imply that loops no longer conserve the lattice orientation of figure 1, indicating that a crucial *symmetry* is broken.

To identify this symmetry we need to get into a bit of algebra. We consider the lattice in figure 1 and a transfer matrix T propagating vertically. We introduce a supersymmetric (SUSY) formulation [14, 15]: each edge carries a $\mathbb{Z}_2$-graded vector space of dimensions

$m+n$ (resp. n) for the even bosonic (resp. odd fermionic) subspace ($m+n$, $n \geqslant 0$ are integers). We label edges $i = 0, 1, \ldots, 2L-1$ for a system of width $2L$. The $\mathbb{Z}_2$ space is chosen as the fundamental $\Box$ of the Lie superalgebra gl$(m+n|n)$ for i even (down arrow), and its dual $\bar{\Box}$ for i odd (up arrow): T acts on the graded tensor product $\mathcal{H} = (\Box \otimes \bar{\Box})^{\otimes L}$. For generic m, the tensor products $\Box \otimes \bar{\Box}$ and $\bar{\Box} \otimes \Box$ decompose as the direct sum of the singlet and the adjoint. The projectors on the singlet obey the Temperley–Lieb algebra relations $E_i^2 = mE_i$, $[E_i, E_j] = 0$ for $|i-j| > 2$, and $E_i E_{i\pm1} E_i = E_i$ (and here $m = 0$). We have $T \equiv T_1 T_3 \cdots T_{2L-3} T_0 T_2 \cdots T_{2L-2}$, where $T_i = 1 + xE_i$. By taking either of the two terms in T_i for each vertex, the expansion of figure 1 is obtained, with a power of x for each vertex, and a factor $(n+m) - n = \mathrm{str}\, 1 = m$ for each loop. The latter equals the supertrace in the fundamental representation (denoted str) of 1, since states in $\mathcal{H}$ flow around the loop. Isotropic dense polymers now correspond to $m = 0$ and $x = 1$. Letting $x \to 0$ allows one to extract the spin chain Hamiltonian $H \propto -\sum_i E_i$ acting on $\mathcal{H}$. The interaction is simply the invariant quadratic coupling (Casimir), providing a natural generalization of the Heisenberg chain to the gl$(n+m|n)$ case.

The first important point now is that allowing four-leg crossings breaks the gl$(n|n)$ symmetry since the products $\Box \otimes \bar{\Box}$ and $\bar{\Box} \otimes \Box$ decompose on only two invariant tensors in gl (the symmetry is actually broken down to the orthosymplectic subgroup).

The second important point is that, for models such as the Heisenberg chain and its generalizations, there is a systematic way to obtain a continuum quantum field theory [15], which is a nonlinear σ-model with target space the symmetry supergroup (here $U(n+m|n)$), modulo the isotropy supergroup of the highest weight state (see [16, 17] for related non-SUSY examples, and [14, 18] for SUSY random fermion problems). Here we obtain $U(n+m|n)/U(1) \times U(n+m-1|n) \cong \mathbf{CP}^{n+m-1|n}$, a SUSY version of complex projective space. Moreover, the mapping shows that this model has a topological angle $\theta = \pi$.

Dense polymers and σ-models. Let us now make things concrete: the fields can be represented by complex components z^a ($a = 1, \ldots, n+m$) and ζ^α ($\alpha = 1, \ldots, n$), where z^a is commuting and ζ^α is anticommuting. In these coordinates, at each point in spacetime, the solutions to the constraint $z_a^\dagger z^a + \zeta_\alpha^\dagger \zeta^\alpha = 1$ (we use the conjugation $\dagger$ that obeys $(\eta\xi)^\dagger = \xi^\dagger \eta^\dagger$ for any η, ξ), modulo $U(1)$ phase transformations $z^a \mapsto \mathrm{e}^{\mathrm{i}B} z^a$, $\zeta^\alpha \mapsto \mathrm{e}^{\mathrm{i}B}\zeta^\alpha$, parametrize $\mathbf{CP}^{n+m-1|n}$. The Lagrangian density in 2D Euclidean spacetime is

$$\mathcal{L} = \frac{1}{2g_\sigma^2}\big[(\partial_\mu - \mathrm{i}a_\mu)z_a^\dagger(\partial_\mu + \mathrm{i}a_\mu)z^a + (\partial_\mu - \mathrm{i}a_\mu)\zeta_\alpha^\dagger(\partial_\mu + \mathrm{i}a_\mu)\zeta^\alpha\big] + \frac{\mathrm{i}\theta}{2\pi}(\partial_\mu a_\nu - \partial_\nu a_\mu), \tag{1}$$

where $a_\mu = \frac{\mathrm{i}}{2}\big[z_a^\dagger \partial_\mu z^a + \zeta_\alpha^\dagger \partial_\mu \zeta^\alpha - \big(\partial z_a^\dagger\big) z^a - \big(\partial \zeta_\alpha^\dagger\big)\zeta^\alpha\big]$ for $\mu = 1, 2$. The fields are subject to the constraint, and under the $U(1)$ gauge invariance a_μ transforms as a gauge potential; a gauge must be fixed in any calculation. This set-up is similar to the non-SUSY $\mathbf{CP}^{m-1}$ model in [19, 20]. The coupling constants are g_σ^2, the usual σ-model coupling (there is only one such coupling because the target supermanifold is a supersymmetric space, and hence the metric on the target space is unique up to a constant factor), and θ, the coefficient of the topological term (θ is defined modulo 2π).

First we note a well-known important point about the SUSY models: the physics is the same for all n, in the following sense. For example, in the present model, correlation functions of operators that are local functions (possibly including derivatives) of components $a \leqslant n_1 + m$, $\alpha \leqslant n_1$ for some n_1 are equal for any $n \geqslant n_1$, due to cancellation of the 'unused' even and odd index values. This can be seen in perturbation theory because the unused index values appear only in summations over closed loops, and their contributions cancel, but is also true nonperturbatively (it can be shown in the lattice constructions we discuss below). In particular, the renormalization group (RG) flow of the coupling g_σ^2 is the same as for $n = 0$, a non-SUSY σ-model. For the case of $\mathbf{CP}^{n+m-1|n}$, the perturbative β-function is the same as for $\mathbf{CP}^{m-1}$, namely (we will not be precise about the normalization of g_σ^2)

$$\frac{\mathrm{d}g_\sigma^2}{\mathrm{d}\xi} = \beta\left(g_\sigma^2\right) = mg_\sigma^4 + O\left(g_\sigma^6\right), \tag{2}$$

where $\xi = \log L$, with L being the length scale at which the coupling is defined (see e.g. [21], equation (3.4)). (The β-function for θ is zero in perturbation theory, and that for g_σ^2 is independent of θ.) For $m > 0$, if the coupling is weak at short length scales, then it flows to larger values at larger length scales. For $\theta \neq \pi$ (mod 2π), the coupling becomes large, the $U(n+m|n)$ symmetry is restored and the theory is massive. However, a transition is expected at $\theta = \pi$ (mod 2π). For $m > 2$, this transition is believed to be of first order, while it is of second order for $m \leqslant 2$ [16]. In the latter case, the system with $\theta = \pi$ flows to a conformally invariant fixed-point theory. At the fixed point, a change in θ is a relevant perturbation that makes the theory massive.

For $m = 0$, the perturbative β-function vanishes identically. This can be seen either from direct calculations, which have been done to at least four-loop order [21], or from an argument similar to that in [9]: for $n = 1$, the σ-model reduces to the massless free-fermion theory [22] $\mathcal{L} \propto \frac{1}{2g_\sigma^2}\partial_\mu\zeta^\dagger\partial_\mu\zeta$ and further the θ-term becomes trivial in this case. Thus, for all σ-model couplings $g_\sigma^2 > 0$, the $n = 1$ theory is non-interacting. The free-fermion theory is conformal with $c = -2$, and θ is a redundant perturbation, as it does not appear in the action (a similar argument appeared in [23]). By the above argument, conformal invariance with $c = -2$ should hold for all n, and also for all g_σ^2 and θ, though the action is no longer non-interacting in general. Thus, the β-function also vanishes non-perturbatively. In general, the scaling dimensions will vary with the coupling g_σ^2, so changing g_σ^2 is an exactly marginal perturbation, though for $n = 1$ the coupling can be scaled away, so there is no dependence on the coupling in the exponents related to those multiplets of operators that survive at $n = 1$. Hence for $n = 1$, the exactly marginal perturbation that changes g_σ^2 is redundant.

Introducing the six-leg crossings. For $n = 1$, the σ-model thus does not exhibit very interesting physics. It also describes very few observables in the dense polymer problem. Indeed, the underlying algebra PSL(1|1) does not admit any non-trivial invariant tensor, so the only ℓ-leg operators present have $\ell = 0, 2$, and they are moreover degenerate—and part of an indecomposable block. These observables are expected to be present in all theories with $n > 1$ as well, and not to depend on the coupling constant g_σ^2. However, for $n > 1$, more observables are possible. For example, ℓ-leg operators for all even ℓ exist, and correspond to fully symmetric invariant tensors of PSL$(n|n)$; there is no reason why the corresponding conformal dimensions should not depend on g_σ^2, and indeed we will shortly see that they do.

For this, we need to be able to tune g_σ^2 in the lattice model. We propose doing so by allowing not four-leg but *six-leg* crossings. This can be described most conveniently by going to the Hamiltonian formalism, and adding interactions that preserve the symmetry. As discussed earlier, four-leg crossings break the symmetry down to the orthosymplectic subgroup. On the other hand, six-leg crossings correspond to exchanging representations at position $i, i+2$ while the one at $i+1$ just goes through, and are perfectly compatible with the gl$(n+m|n)$ symmetry. The Hamiltonian then becomes

$$H \propto -\sum_i (E_i + wP_{i,i+2}). \tag{3}$$

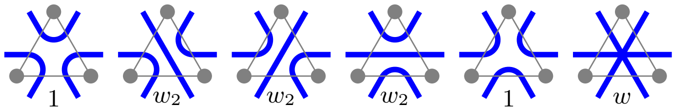

Figure 2. Vertices and weights for dense polymers on the triangular lattice. When $w = 0$, this is equivalent [24] to a Potts model with spins on the circles and arbitrary interactions within the gray triangles.

Our first claim is that the continuum limit of (3) is described by the superprojective σ-model $\mathbf{CP}^{n-1|n}$ with the coupling $g_\sigma^2(w)$, at $\theta = \pi$. Note that we could more generally study the spectrum of the Hamiltonian $H \propto -\sum_i [E_i + w P_{i,i+2} + w_2(E_i E_{i+1} + E_{i+1} E_i)]$. The symmetries are unchanged, and one expects the continuum limit to be described by the same σ-model, with now $g_\sigma^2(w, w_2)$. This is confirmed by numerical calculations. Finally, a more pleasant realization of the same physics is provided by a model of dense polymers on the triangular lattice, where six-leg crossings can naturally take place; see figure 2. We shall call the Boltzmann weight of these vertices w as well, and the same conclusions will hold for this model as for the spin chain (3).

We first check what happens for $n = 1$, where everything can be reformulated in terms of free-fermion operators and their adjoints [12] $f_i, f_i^\dagger$, obeying $\{f_i, f_{i'}\} = 0$, $\{f_i, f_{i'}^\dagger\} = (-1)^i \delta_{i'}$ through $E_i = (f_i^\dagger + f_{i+1}^\dagger)(f_i + f_{i+1})$ and $P_{i,i+2} = (-1)^i + (f_{i-1}^\dagger - f_{i+1}^\dagger)(f_{i-1} - f_{i+1})$. Since both are quadratic, it is easy to show that the continuum limit of (3) is unchanged, with w only affecting the sound velocity and the fine structure of the Jordan blocks.

One can easily argue that the ground-state energy is the same for the $n = 1$ and $n > 1$ models, whence $c = -2$ independently of w. This is confirmed by transfer matrix calculations for the model in figure 2. The $\ell = 2$ exponent is conjugate to the fractal dimension of the loop, hence zero.

A numerical study of the $\ell > 2$ leg exponents then clearly shows that they are non-trivial, decreasing functions of w. To discuss this in some more detail we place ourselves in the simplest case of free boundary conditions. The exponents at the special point $w = 0$ are well known to be $h_\ell^0 = h_{1,1+\ell} = \frac{\ell(\ell-2)}{8}$. We next assume that $w \to \infty$ corresponds to the weak-coupling limit of the σ-model, $g_\sigma^2 \to 0$. This is qualitatively very reasonable: in the limit of large w, the system almost splits into two subsystems with the gl$(n|n)$ symmetry involving only the fundamental or only its dual, with in both cases a simple interaction of the type $P_{i,i+1}$. Such models are well known to be integrable, and their physics to be described by a weak-coupling limit not unlike the XXX ferromagnetic spin chain. In such a limit, we can analyze the spectrum using the minisuperspace approach, that is, by analyzing quantum mechanics on the target manifold. The spectrum of the Laplacian on the ordinary projective space $\mathbf{CP}^{m-1} = U(m)/U(1) \times U(m-1)$ is well known to be of the form $E_l \propto 4l(l + m - 1)$, so, setting $m = 0$, we find that [25] $h_l^{\text{wc}} = g_\sigma^2 \frac{l(l-1)}{2}$. Here l in an integer, which we can identify using the PSL$(n|n)$ representation theory with $\ell/2$. Remarkably, h_l^{wc} coincides with the known result h_ℓ^0 at $w = 0$ (ordinary dense polymers) if we identify $g_\sigma^2 = 1$ in that case.

Conjecture for the exact exponents. We conjecture that the boundary conformal dimensions in our model are simply linear in the Casimir of the associated representation of PSL$(n|n)$. This is due to the structure of the perturbation theory where the vanishing of the dual Coxeter number—the Casimir in the adjoint—suggests exactness of the minisuperspace approximation (see [26] for a related case). A more thorough study of this perturbation theory, together with non-perturbative arguments, will appear elsewhere [27]. For now, we simply propose that the

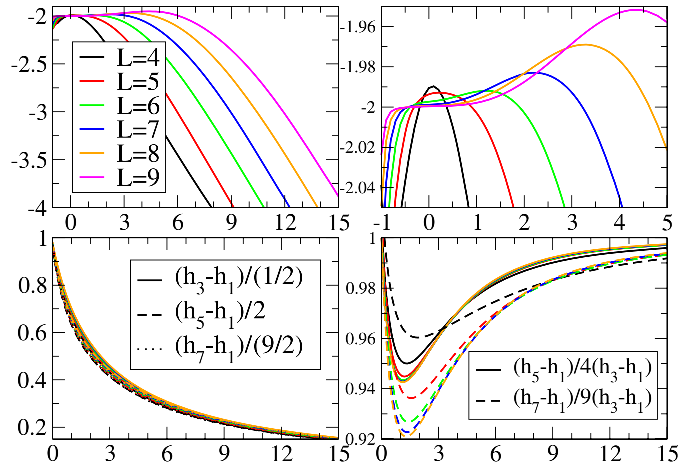

Figure 3. The two upper panels show the central charge as a function of the intersection weight w for width L strips of the triangular lattice. In the text we show analytically that $c = -2$ always. The lower left panel represents the effective coupling constant g_σ^2 extracted from (4) using different values of ℓ. The collapse on a single curve is quite striking. The lower right panel shows details of the exponents, in particular the region close to $w = 0$ where convergence appears actually less good than in the previous curves.

exponents be given by

$$h_\ell = g_\sigma^2 \frac{\ell(\ell-2)}{8}, \tag{4}$$

where g_σ^2 is a decreasing function of w, equal to unity when $w = 0$, and vanishing at large w.

This conjecture is compared with the results of exact diagonalizations in figure 3 (lower panels), where we have represented the function $g_\sigma^2(w)$ as extracted from (4) and various ℓ. The different estimates collapse on a single curve over the whole range of w values, in agreement with the conjecture.

For the model of figure 2 it is technically difficult to study operators with ℓ even. The σ-model formalism can however be extended to ℓ odd, and the arguments leading to (4) extended to this case [27]. Exact diagonalization of the spin chain Hamiltonian on $2L = 18$ sites yields results for ℓ even which look like the lower left panel of figure 3, except that w now has a different meaning. The sound velocity is determined from analytical results for the $n = 1$ case.

The case of periodic boundary conditions seems to be quite different. In this case, the known values of the *bulk* polymer exponents at $w = 0$ are $h_\ell = \frac{\ell^2-4}{32} = \frac{l^2-1}{8}$. The fact that $h_6 = 1$ provides an independent argument for the marginality of the w perturbation. Meanwhile, note that h_ℓ now do *not* have the minisuperspace form. For large w, one can however argue that the minisuperspace form remains valid, as is confirmed numerically. This suggests again that maybe the point $w = 0$ is singular [28] (there are indications of this in the fine structure of the boundary spectrum as well). It could also be that in the periodic case, the arguments that the minisuperspace should be exact for any w fail, which agrees with the expectations for a related model in [26]. More work is needed to clarify this point.

We checked that, within numerical accuracy, staggering the chain produces similar results but with a coupling constant that now depends on w and the staggering parameter—that is, the θ angle in the continuum limit.

In conclusion, we have shown that allowing intersections where three lines cross profoundly modifies the dense polymer problem. It gives rise to a critical line of conformal field theories, with the central charge $c = -2$, which can be identified with the long-distance limit of a conformal σ-model such as those studied in the AdS/CFT correspondence. Our identification leads moreover to the proposal of an exact formula (4) for the ℓ-leg polymer exponents in the boundary case, and opens the way to tackling the σ-model using lattice techniques.

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References

[1] Nienhuis B 1984 *J. Stat. Phys.* **34** 731
[2] Freedman M 2003 *Commun. Math. Phys.* **234** 129
[3] Di Francesco P, Mathieu P and Sénéchal D 1996 *Conformal Field Theory* (Berlin: Springer)
[4] Cardy J 2005 *Ann. Phys.* **318** 81
[5] Werner W 2000 arXiv:math/0007042
[6] Petschek R G and Pfeuty P 1987 *Phys. Rev. Lett.* **58** 1096
[7] Jacobsen J L, Read N and Saleur H 2003 *Phys. Rev. Lett.* **90** 090601
[8] Parisi G and Sourlas N 1980 *J. Phys. Lett. (Paris)* **41** L403
[9] Berkovits N, Vafa C and Witten E 1999 *J. High Energy Phys.* JHEP03(1999)018
[10] Bershadsky M, Zhukov S and Vaintrob A 1999 *Nucl. Phys.* B **559** 205
[11] Gotz G, Quella T and Schomerus V 2007 *J. High Energy Phys.* JHEP03(2007)003
[12] Read N and Saleur H 2007 *Nucl. Phys.* B **777** 263
[13] Read N and Saleur H 2007 *Nucl. Phys.* B **777** 316
[14] Gruzberg I A, Ludwig A W W and Read N 1999 *Phys. Rev. Lett.* **82** 4524
[15] Read N and Saleur H 2001 *Nucl. Phys.* B **613** 409
[16] Affleck I 1985 *Nucl. Phys.* B **257** 397
[17] Read N and Sachdev S 1989 *Nucl. Phys.* B **316** 609
[18] Zirnbauer M R 1994 *Ann. Phys. (Berlin)* **3** 513
[19] D'Adda A, Lüscher M and Di Vecchia P 1978 *Nucl. Phys.* B **146** 63
[20] Witten E 1979 *Nucl. Phys.* B **149** 285
[21] Wegner F 1989 *Nucl. Phys.* B **316** 663
[22] Kausch H G 1995 arXiv:hep-th/9510149
[23] Bocquet M, Serban D and Zirnbauer M R 2000 *Nucl. Phys.* B **578** 628
[24] Wu F Y and Lin K Y 1980 *J. Phys. A: Math. Gen.* **13** 629
[25] Zhang R B and Zou Y M 2005 *J. Math. Phys.* **46** 043513
[26] Quella T, Schomerus V and Creutzig T 2008 *J. High Energy Phys.* JHEP10(2008)024
[27] Candu C, Mitev V, Quella T, Saleur H and Schomerus V 2010 *J. High Energy Phys.* JHEP02(2010)015 (arXiv:0908.0878)
[28] Candu C and Saleur H 2009 *Nucl. Phys.* B **808** 487